GLOBAL ATTRACTORS FOR THE ONE DIMENSIONAL WAVE EQUATION WITH DISPLACEMENT DEPENDENT DAMPING

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Abstract. We study the long-time behavior of solutions of the one dimensional wave equation with nonlinear damping coefficient. We prove that if the damping coefficient function is strictly positive near the origin then this equation possesses a global attractor.

1. INTRODUCTION

In this paper, we consider the following Cauchy problem:

$$u_{tt} + \sigma(u)u_t - u_{xx} + \lambda u + f(u) = g(x), \quad (t,x) \in (0,\infty) \times R, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad x \in R, \tag{1.2}$$

where λ is a positive constant, $g \in L_1(R) + L_2(R)$ and nonlinear functions $f(\cdot)$ and $\sigma(\cdot)$ satisfy the following conditions:

$$f \in C^1(R), \quad f(u)u \geq 0, \quad \forall\ u \in R, \tag{1.3}$$

$$\sigma \in C(R), \quad \sigma(0) > 0, \quad \sigma(u) \geq 0, \quad \forall\ u \in R. \tag{1.4}$$

Applying standard Galerkin's method and using techniques of [6, Proposition 2.2], it is easy to prove the following existence and uniqueness theorem:

Theorem 1. *Assume that the conditions (1.3)-(1.4) hold. Then for any $T > 0$ and $(u_0, u_1) \in \mathcal{H} := H^1(R) \times L_2(R)$ the problem (1.1)-(1.2) has a unique weak solution $u \in C([0,T]; H^1(R)) \cap C^1([0,T]; L_2(R)) \cap C^2([0,T]; H^{-1}(R))$ on $[0,T] \times R$ such that*

$$\|(u(t), u_t(t))\|_{\mathcal{H}} \leq c(\|(u_0,u_1)\|_{\mathcal{H}}), \quad \forall\, t \geq 0,$$

where $c : R_+ \to R_+$ is a nondecreasing function. Moreover if $v \in C([0,T]; H^1(R)) \cap C^1([0,T]; L_2(R)) \cap C^2([0,T]; H^{-1}(R))$ is also weak solution to (1.1)-(1.2) with initial data $(v_0, v_1) \in \mathcal{H}$, then

$$\|u(t) - v(t)\|_{L_2(R)} + \|u_t(t) - v_t(t)\|_{H^{-1}(R)} \leq$$

$$\leq \widetilde{c}(T, \widetilde{R}) \left(\|u_0 - v_0\|_{L_2(R)} + \|u_1 - v_1\|_{H^{-1}(R)} \right), \quad \forall\, t \in [0,T],$$

where $\widetilde{c} : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $\widetilde{R} = \max\{\|(u_0,u_1)\|_{\mathcal{H}}, \|(v_0,v_1)\|_{\mathcal{H}}\}$.

Thus, by the formula $(u(t), u_t(t)) = S(t)(u_0, u_1)$, the problem (1.1)-(1.2) generates a weak continuous semigroup $\{S(t)\}_{t \geq 0}$ in $\mathcal{H}$, where $u(t,x)$ is a weak solution of (1.1)-(1.2), determined by Theorem 1.1, with initial data (u_0, u_1).

The attractors for equation (1.1) in the finite interval were studied in [2], assuming positivity of $\sigma(\cdot)$. For two dimensional case, the attractors for the wave equation with displacement dependent damping were investigated in [7] under conditions

$$\sigma \in C^1(R),\ 0 < \sigma_0 \leq \sigma(u) \leq c(1 + |u|^q), \quad \forall\, u \in R,\ 0 \leq q < \infty,$$

and

$$|\sigma'(u)| \leq c[\sigma(u)]^{1-\varepsilon}, \quad \forall\, u \in R, \quad 0 < \varepsilon < 1, \tag{1.5}$$

on the damping coefficient. Recently, in [3], condition (1.5) has been improved as

$$|\sigma'(u)| \leq c\sigma(u), \ \forall\, u \in R.$$

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For the three dimensional bounded domain case, the existence of a global attractor for the wave equation with displacement dependent damping was proved in [6] when $\sigma(\cdot)$ is a strictly positive and globally bounded function. In this case, when $\sigma(\cdot)$ is not globally bounded, but is equal to a positive constant in a large enough interval, the existence of a global attractor has been established in [4].

In the articles mentioned above, the existence of global attractors was proved under positivity or strict positivity condition on the damping coefficient function $\sigma(\cdot)$. In this paper, we study a global attractor for (1.1)-(1.2) under weaker conditions on $\sigma(\cdot)$ and prove the following theorem:

Theorem 2. *Under conditions (1.3)-(1.4) a semigroup $\{S(t)\}_{t\geq 0}$ generated by (1.1)-(1.2) possesses a global attractor in $\mathcal{H}$.*

2. Proof of Theorem 1.2

To prove this theorem we need the following lemma:

Lemma 1. *Let conditions (1.3)-(1.4) hold and let B be a bounded subset of $\mathcal{H}$. Then for any $\varepsilon > 0$ there exist $T_0 = T_0(\varepsilon, B) > 0$ and $r_0 = r_0(\varepsilon, B) > 0$ such that*

$$\|S(t)\varphi\|_{H^1(R\backslash(-r_0,r_0))\times L_2(R\backslash(-r_0,r_0))} < \varepsilon, \ \ \forall\, t \geq T_0, \quad \forall\, \varphi \in B. \tag{2.1}$$

Proof. Let $(u_0, u_1) \in B$ and $S(t)(u_0, u_1) = (u(t), u_t(t))$. Multiplying (1.1) by u_t and integrating over $(0, t) \times R$ we obtain

$$\|u_t(t)\|^2_{L_2(R)} + \|u(t)\|^2_{H^1(R)} + \int_0^t\int_R \sigma(u(\tau, x))u_t^2(\tau, x)dxd\tau \leq c_1, \qquad \forall\, t \geq 0. \tag{2.2}$$

Let $\eta \in C^1(R)$, $0 \leq \eta(x) \leq 1$, $\eta(x) = \begin{cases} 0, & |x| \leq 1 \\ 1, & |x| \geq 2 \end{cases}$, $\eta_r(x) = \eta(\frac{x}{r})$ and $\Sigma(u) = \int_0^u \sigma(s)ds$. Multiplying (1.1) by $\eta_r^2\Sigma(u)$, integrating over $(0, t) \times R$ and taking into account (2.2) we have

$$\int_0^t\int_R \eta_r^2(x)\sigma(u(\tau, x))u_x^2(\tau, x)dxd\tau + \lambda\int_0^t\int_R \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)dxd\tau \leq$$

$$\leq c_2(1 + \sqrt{t} + \frac{t}{r} + t\,\|g\|_{L_1(R\backslash(-r,r))+L_2(R\backslash(-r,r))}), \qquad \forall\, t \geq 0, \qquad \forall\, r > 0. \tag{2.3}$$

By (1.4), there exists $l > 0$, such that

$$\frac{\sigma(0)}{2} \leq \sigma(s) \leq 2\sigma(0), \qquad \forall\, s \in [-l, l]. \tag{2.4}$$

Using embedding $H^{\frac{1}{2}+\varepsilon}(R) \subset L_\infty(R)$ and taking into account (2.2) and (2.4) we find

$$\int_0^t\int_R \eta_r^2(x)u^2(\tau, x)dxd\tau \leq \frac{2}{\sigma(0)}\int_0^t \int_{\{x:|u(\tau,x)|\leq l\}} \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)dxd\tau +$$

$$+c_3\int_0^t \int_{\{x:|u(\tau,x)|>l\}} \eta_r^2(x)\,|u(\tau, x)|\,dxd\tau \leq \frac{2}{\sigma(0)}\int_0^t \int_{\{x:|u(\tau,x)|\leq l\}} \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)dxd\tau +$$

$$+\frac{2c_3}{\sigma(0)l}\int_0^t \int_{\{x:|u(\tau,x)|>l\}} \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)dxd\tau$$

and consequently

$$\int_0^t \|\eta_r u(\tau)\|^5_{L_\infty(R)}\,d\tau \leq c_4\int_0^t \|\eta_r u(\tau)\|^2_{L_2(R)}\,d\tau \leq c_5\int_0^t\int_R \eta_r^2(x)\Sigma(u(\tau, x))u(\tau, x)dxd\tau, \tag{2.5}$$

for $r \geq 1$. So by (2.2), (2.3) and (2.5), we get

$$\int_0^t \Big[\left\|\eta_{2r}\sigma^{\frac{1}{2}}(u(\tau))u_t(\tau)\right\|^2_{L_2(R)} + \left\|\eta_{2r}\sigma^{\frac{1}{2}}(u(\tau))u_x(\tau)\right\|^2_{L_2(R)} + \lambda\left\|\eta_{2r}\sigma^{\frac{1}{2}}(u(\tau))u(\tau)\right\|^2_{L_2(R)} +$$

$$+ \left\|\eta_r u(\tau)\right\|^5_{L_\infty(R)}\Big] d\tau \leq c_6(1+\sqrt{t}+\frac{t}{r}+t\left\|g\right\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))}), \quad \forall\, t \geq 0, \ \forall\, r \geq 1. \quad (2.6)$$

Now denote $\Phi_r(u(t)) := \frac{1}{2}\left\|\eta_r u_t(t)\right\|^2_{L_2(R)} + \frac{1}{2}\left\|\eta_r u_x(t)\right\|^2_{L_2(R)} + \mu\left\langle \eta_r u_t(t),\ \eta_r u(t)\right\rangle + \frac{\lambda}{2}\left\|\eta_r u(t)\right\|^2_{L_2(R)} + \left\langle \eta_r F(u(t)),\ \eta_r\right\rangle + \left\langle \eta_r g,\ \eta_r u(t)\right\rangle$, where $\mu = \min\left\{\sqrt{\frac{\lambda}{2}}, \frac{\sigma(0)}{5}, \frac{\lambda}{2\sigma(0)}\right\}$, $\langle u,\ v\rangle = \int_R u(x)v(x)dx$ and $F(u) = \int_0^u f(s)ds$. By (2.4) and (2.6), it follows that for any $\delta > 0$ there exist $T_\delta = T_\delta(B) > 0$, $r_{1,\delta} = r_{1,\delta}(B) > 1$ and for any $r \geq r_{1,\delta}$ there exists $t^*_{\delta,r} \in [0, T_\delta]$ such that

$$\Phi_r(u(t^*_{\delta,r})) < \delta, \quad \forall\, r \geq r_{1,\delta}. \quad (2.7)$$

Again by (2.2), we have

$$\left\|\eta_r u(t)\right\|_{L_2(R)} \leq \left\|\eta_r u(t^*_{\delta,r})\right\|_{L_2(R)} + \int_{t^*_{\delta,r}}^t \left\|\eta_r u_t(s)\right\|_{L_2(R)} ds \leq \left\|\eta_r u(t^*_{\delta,r})\right\|_{L_2(R)} + c_7(t - t^*_{\delta,r})$$

and consequently

$$\left\|\eta_r u(t)\right\|^3_{L_\infty(R)} \leq c_8\left\|\eta_r u(t)\right\|_{L_2(R)} \leq c_9(\Phi_r^{\frac{1}{2}}(u(t^*_{\delta,r})) + \left\|g\right\|^{\frac{1}{2}}_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))} + t - t^*_{\delta,r}) <$$

$$< c_9(\delta^{\frac{1}{2}} + \left\|g\right\|^{\frac{1}{2}}_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))} + t - t^*_{\delta,r}), \ \forall\, t \geq t^*_{\delta,r}, \ \forall\, r \geq r_{1,\delta}.$$

Denoting $T^*_{\delta,r} = t^*_{\delta,r} + \frac{l^3}{3c_9}$ and choosing $\delta \in (0, \frac{l^6}{9c_9^2})$, by the last inequality, we can say that there exists $r_{2,\delta} \geq 2r_{1,\delta}$ such that

$$\left\|u(t)\right\|_{L_\infty(R\setminus(-r_{2,\delta},r_{2,\delta}))} < l, \quad \forall\, t \in [t^*_{\delta,r}, T^*_{\delta,r}]. \quad (2.8)$$

Now multiplying (1.1) by $\eta_r^2(u_t + \mu u)$, integrating over R and taking into account (2.4) and (2.8) we obtain

$$\frac{d}{dt}\Phi_r(u(t)) + c_{10}\Phi_r(u(t)) \leq c_{11}(\frac{1}{r} + \left\|g\right\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))}), \ \forall\, t \in [t^*_{\delta,r}, T^*_{\delta,r}],$$

and consequently

$$\Phi_r(u(t)) \leq \Phi_r(u(t^*_{\delta,r}))e^{-c_{10}(t-t^*_{\delta,r})} + c_{11}(\frac{1}{r} + \left\|g\right\|_{L_1(R\setminus(-r,r))+L_2(R\setminus(-r,r))})\frac{1-e^{-c_{10}(t-t^*_{\delta,r})}}{c_{10}}, \quad (2.9)$$

for $r \geq r_{2,\delta}$. By (2.7) and (2.9), there exists $r_{3,\delta} \geq r_{2,\delta}$ such that

$$\Phi_r(u(t)) < \delta, \quad \forall\, r \geq r_{3,\delta}, \ \forall\, t \in [t^*_{\delta,r}, T^*_{\delta,r}].$$

Hence denoting by n_δ the smallest integer number which is not less than $\frac{3c_9T_\delta}{l^3}$ and applying above procedure at most n_δ time, we find

$$\Phi_r(u(T_\delta)) < \delta, \quad \forall\, r \geq r_{4,\delta},$$

for some $r_{4,\delta} \geq 2^{n_\delta}r_{1,\delta}$. From the last inequality it follows that for any $\varepsilon > 0$ there exist $\widehat{T}_\varepsilon = \widehat{T}_\varepsilon(B) > 0$ and $\widehat{r}_\varepsilon = \widehat{r}_\varepsilon(B) > 0$ such that

$$\left\|S(\widehat{T}_\varepsilon)\varphi\right\|_{H^1(R\setminus(-\widehat{r}_\varepsilon,\widehat{r}_\varepsilon))\times L_2(R\setminus(-\widehat{r}_\varepsilon,\widehat{r}_\varepsilon))} < \varepsilon, \quad \forall\, \varphi \in B.$$

Since, by (2.2), $B_0 = \bigcup_{t\geq 0} S(t)B$ is a bounded subset of $\mathcal{H}$, for any $\varepsilon > 0$ there exist $T_0 = T_0(\varepsilon, B) > 0$ and $r_0 = r_0(\varepsilon, B) > 0$ such that

$$\left\|S(T_0)\varphi\right\|_{H^1(R\setminus(-r_0,r_0))\times L_2(R\setminus(-r_0,r_0))} < \varepsilon, \quad \forall\, \varphi \in B_0.$$

Taking into account positively invariance of B_0, from the last inequality we obtain (2.1). $\square$

By (2.1) and (2.4), for any bounded subset B of $\mathcal{H}$ there exist $\widehat{T}_0 = \widehat{T}_0(B) > 0$ and $\widehat{r}_0 = \widehat{r}_0(B) > 0$ such that

$$\sigma(u(t,x)) \geq \frac{\sigma(0)}{2}, \qquad \forall\, t \geq \widehat{T}_0, \quad \forall\, |x| \geq \widehat{r}_0. \tag{2.10}$$

Hence using techniques of [5] one can prove the asymptotic compactness of the semigroup $\{S(t)\}_{t\geq 0}$, which is included in the following lemma:

Lemma 2. *Assume that conditions (1.3)-(1.4) hold and B is bounded subset of $\mathcal{H}$. Then every sequence of the form $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$, $\{\varphi_n\}_{n=1}^{\infty} \subset B$, $t_n \to \infty$, has a convergent subsequence in $\mathcal{H}$.*

By (2.10) and the unique continuation result of [8], it is easy to see that problem (1.1)-(1.2) has a strict Lyapunov function (see [1] for definition). Thus according to [1, Corollary 2.29] the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor.

Remark 1. *We note that, for the problem considered in* [2], *from compact embedding $H_0^1(0,\pi) \subset C[0,\pi]$, it immediately follows that $\sigma(u(t,x)) \geq \frac{\sigma(0)}{2}$, $\forall\, t \geq 0$, $\forall\, x \in [0,\varepsilon] \cup [\pi-\varepsilon,\pi]$ for some $\varepsilon \in (0,\pi)$. So a global attractor still exists if one replaces the positivity condition on $\sigma(\cdot)$ by the $\sigma(0) > 0$.*

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